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Isometries of Matrix Algebras*

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The sum of the first k singular values of an n-square complex matrix is a norm, $1 \leqslant k \leqslant n$. In this paper all isometries of the algebra of n-square complex matrices which preserve this norm are determined. For $k = 1$ and $k = n$ the result specializes to earlier work of Morita and Russo, respectively.

1. INTRODUCTION

Thirty-five years ago Morita [9] proved that if T is a linear map on the space of $m \times n$ complex matrices $M_{m,n}(\mathbb{C})$ to itself, which holds fixed the maximum singular value (i.e., the Hilbert norm) of every matrix in $M_{m,n}(\mathbb{C})$, then

$$\begin{aligned} & T(A) = UAV \qquad \text{for all } A, \\ \text{or (if } m = n), \quad & \\ & T(A) = UA^tV \qquad \text{for all } A, \end{aligned} \tag{1}$$

where U and V are fixed unitary matrices. In [11], Russo proved that if $m = n$ and T preserves the sum of all the singular values of every $A \in M_{n,n}(\mathbb{C})$ then T has the form (1). Russo's proof depends on showing that T holds the unitary group invariant and then using an earlier result of one of the present authors [4]. In a recent Ph. D. thesis, Arazy [1] obtained a result which in the finite-dimensional case implies that any linear map on $M_{n,n}(\mathbb{C})$ holding fixed

$$\left(\sum_{j=1}^{n} \alpha_j(A)^p\right)^{1/p}, \qquad 1 \leqslant p \leqslant \infty, \quad p \neq 2, \tag{2}$$

($\alpha_1(A) \geqslant \alpha_2(A) \geqslant \cdots \geqslant \alpha_n(A)$ are the singular values of A) has the form (1).

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In [5] it is proved as a corollary of a more general result that if $0 < p \leqslant 1$ and T holds the function (2) invariant, then T also has the form (1). In [6] similar results were obtained for linear maps holding fixed the elementary symmetric functions of the squares of the singular values. We assume henceforth that $m = n$ and we write $M_{n,n}(\mathbb{C})$ as $M_n(\mathbb{C})$.

Let $1 \leqslant k \leqslant n$ and define

$$\varphi_{p,k}(A) = \left(\sum_{j=1}^{k} \alpha_j(A)^p\right)^{1/p}, \qquad 1 \leqslant p \leqslant \infty. \tag{3}$$

The function $\varphi_{p,k}\colon M_n(\mathbb{C}) \to [0, \infty)$ is a unitarily invariant norm—this follows directly from the von Neumann characterization of such norms in terms of symmetric gauge functions [10]. Thus any linear T satisfying

$$\varphi_{p,k}(T(A)) = \varphi_{p,k}(A) \tag{4}$$

is an isometry of $M_n(\mathbb{C})$ and hence is invertible. The purpose of the present paper is to investigate the structure of the extreme points of the unit sphere in $M_n(\mathbb{C})$ with respect to the norm (3) for $p = 1$ and any k, $1 \leqslant k \leqslant n$, and thereby determine the structure of the isometries T that satisfy (4) for $p = 1$. For $p = 1$ we write $\varphi_{p,k} = \varphi_k$. Thus our main result is:

THEOREM 1. *If $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map and $1 \leqslant k \leqslant n$ is a fixed integer for which*

$$\varphi_k(T(A)) = \varphi_k(A), \qquad A \in M_n(\mathbb{C}),$$

then T has the form (1).

Of course, $k = 1$ and $k = n$ are the specializations of Theorem 1 to the results of Morita and Russo, respectively.

2. EXTREME POINTS

The following extremal characterization of $\varphi_{p,k}(A)$ is useful [3, 8].

LEMMA 1. *If $1 \leqslant p \leqslant \infty$ and $1 \leqslant k \leqslant n$, then*

$$\varphi_{p,k}(A) = \max_{x_t, U} \left\{\sum_{i=1}^{k} |(UAx_t, x_t)|^p\right\}^{1/p} \tag{5}$$

where the maximum in (5) *is over all* o.n. *sets $x_1, \ldots, x_k$ in $\mathbb{C}^n$ and all unitary $U \in M_n(\mathbb{C})$.*

Proof. By the polar factorization theorem we can replace A in (5) by $H = (A^*A)^{1/2} \geqslant 0$. Then

$$\left\{\sum_{t=1}^{k} |(UHx_t, x_t)|^p\right\}^{1/p} = \left\{\sum_{t=1}^{k} |(H^{1/2}x_t, H^{1/2}U^*x_t)|^p\right\}^{1/p}$$
$$\leqslant \left\{\sum_{t=1}^{k} \| H^{1/2}x_t \|^p \| H^{1/2}U^*x_t \|^p\right\}^{1/p}$$
$$\leqslant \left\{\sum_{t=1}^{k} (Hx_t, x_t)^p\right\}^{1/p} \left\{\sum_{t=1}^{k} (HU^*x_t, U^*x_t)^p\right\}. \qquad (6)$$

Let $u_1, \ldots, u_n$ be an o.n. basis of eigenvectors of H corresponding to $\alpha_1, \ldots, \alpha_n$, respectively. Then

$$(Hx_t, x_t) = \sum_{j=1}^{n} s_{tj}\alpha_j, \qquad t = 1, \ldots, k$$

where $S = [s_{tj}]$ is an n-square doubly stochastic matrix. But then

$$\left\{\sum_{t=1}^{k} (Hx_t, x_t)^p\right\}^{1/p} = \left\{\sum_{t=1}^{k} \left(\sum_{j=1}^{n} s_{tj}\alpha_j\right)^p\right\}^{1/p} \qquad (7)$$

and by the Minkowski inequality the function on the right in (7) is convex as a function of S. The vertices of the convex polyhedron of doubly stochastic matrices are precisely the permutation matrices [2] so that the maximum value of (7) is clearly

$$\left\{\sum_{t=1}^{k} \alpha_t{}^p\right\}^{1/p}.$$

The same argument can be used on the second factor in (6). Clearly, $\varphi_{p,k}(A)$ is an achievable value of $\{\sum_{i=1}^{k} |(UAx_t, x_t)|^p\}^{1/p}$. ∎

The characterization (5) in fact provides a direct method of showing that $\varphi_{p,k}(A)$ is a norm on $M_n(\mathbb{C})$. Note that if the rank of A is 1, i.e., $\rho(A) = 1$, then $\varphi_{p,k}(A) = \alpha_1(A)$ is independent of both p and k. Also if $p = 2$, $k = n$, then $\varphi_{2,n}(A)$ is the Euclidean norm induced by the inner product

$$(X, Y) = \operatorname{tr}(Y^*X)$$

on $M_n(\mathbb{C})$. Thus equality holds for nonzero X and Y in the triangle inequality,

$$\varphi_{2,n}(X + Y) \leqslant \varphi_{2,n}(X) + \varphi_{2,n}(Y),$$

iff $Y = cX$, $c > 0$.

There are essentially two steps involved in proving Theorem 1. We first determine the structure of the extreme points of S_k, the unit sphere in $M_n(\mathbb{C})$ consisting of all A for which $\varphi_k(A) \leqslant 1$. Since T is an isometry these are mapped into extreme points. We can then determine the structure of T by referring to known results on linear maps that hold fixed prescribed sets of matrices.

THEOREM 2. *Assume that* $1 \leqslant k < n$; *let* Δ *be an upper triangular matrix with zero main diagonal, and let* D *be a diagonal matrix. Then*

(i) $\varphi_k(\alpha I_n + \Delta) \geqslant k \mid \alpha \mid$ *with equality iff* $\Delta = 0$;

(ii) *if* $\varphi_k(D) = \varphi_k(2I_n - D) = k$, *then* $D = I_n$.

Proof. (i) We can assume $\alpha = 1$ (the result is trivial if $\alpha = 0$). Let $e_t = (\delta_{t1}, \ldots, \delta_{tn})$, $t = 1, \ldots, n$. From Lemma 1

$$\begin{aligned}\varphi_k(I_n + \Delta) &\geqslant \sum_{t=1}^{k} |((I_n + \Delta)e_t, e_t)| \\ &= k.\end{aligned}$$

If some $\Delta_{ij} \neq 0$, $i \neq j$, let $x_1 = (e_i + e_j)/2^{1/2}$, and let $x_2, \ldots, x_k$ be chosen from $e_1, \ldots, e_n$ excluding e_i and e_j. Let U be a diagonal unitary matrix for which $(U^*\Delta U)_{ij} = \mid \Delta_{ij} \mid$. Then from Lemma 1 again

$$\begin{aligned}\varphi_k(I_n + \Delta) &= \varphi_k(I_n + U^*\Delta U) \\ &\geqslant \sum_{t=1}^{k} |((I_n + U^*\Delta U)x_t, x_t)| \\ &= \mid 1 + (U^*\Delta U x_1, x_1)| + k - 1 \\ &= k + \mid \Delta_{ij} \mid /2.\end{aligned}$$

(ii) Write $D = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ in which we can assume $\mid \alpha_1 \mid \geqslant \cdots \geqslant \mid \alpha_n \mid$. Suppose $\mid \alpha_1 \mid > \mid \alpha_n \mid$. Then since $\varphi_k(D) = \mid \alpha_1 \mid + \cdots + \mid \alpha_k \mid$ is the largest sum of k of the $\mid \alpha_i \mid$,

$$\mid \alpha_{n-k+1} \mid + \cdots + \mid \alpha_n \mid < k. \tag{8}$$

Then from (8) and Lemma 1

$$\begin{aligned}\varphi_k(2I_n - D) &\geqslant \left| \sum_{t=n-k+1}^{n} ((2I_n - D)e_t, e_t) \right| \\ &= \mid 2k - (\alpha_{n-k+1} + \cdots + \alpha_n)| \\ &\geqslant 2k - (\mid \alpha_{n-k+1} \mid + \cdots + \mid \alpha_n \mid) \\ &> k.\end{aligned}$$

Since $\varphi_k(2I_n - D) = k$ we conclude that

$$|\alpha_1| = \cdots = |\alpha_n| = 1. \tag{9}$$

The same argument shows that the main diagonal entries of $2I_n - D$ have the same modulus (i.e., let $2I_n - D$ play the role of D, $D = 2I_n - (2I_n - D)$). Thus

$$|2 - \alpha_1| = \cdots = |2 - \alpha_n| = 1. \tag{10}$$

But (9) and (10) together imply that $\alpha_1 = \cdots = \alpha_n = 1$ so that $D = I_n$. ∎

THEOREM 3. *Assume $1 \leqslant k < n$ and let U be a unitary matrix. Then $k^{-1}U$ is an extreme point of S_k. If $k = n$ then $n^{-1}U$ is not an extreme point of S_n.*

Proof. We may assume $U = I_n$. Suppose $I_n = (A + B)/2$, $\varphi_k(A) \leqslant k$, $\varphi_k(B) \leqslant k$. Then the triangle inequality implies that $\varphi_k(A) = \varphi_k(B) = k$. Since $B = 2I_n - A$, A and B commute, and thus they can be simultaneously brought to upper triangular form with a unitary similarity. Hence, we may assume

$$A = D + \Delta$$

and

$$B = (2I_n - D) - \Delta,$$

where D is a diagonal matrix and Δ is strictly upper triangular. From Theorem 2(ii)

$$k = \varphi_k(A) = \varphi_k(D + \Delta) \geqslant \varphi_k(D) \tag{11}$$

and

$$k = \varphi_k(B) = \varphi_k((2I_n - D) - \Delta) \geqslant \varphi_k(2I_n - D). \tag{12}$$

But (11) and (12) imply, by the triangle inequality,

$$\begin{aligned} 2k = \varphi_k(2I_n) &\leqslant \varphi_k(D) + \varphi_k(2I_n - D) \\ &\leqslant 2k. \end{aligned}$$

Thus both (11) and (12) must be equalities. By Theorem 2(ii), $D = I_n$ and hence (11) becomes

$$\varphi_k(I_n + \Delta) = k.$$

From Theorem 2(i), $\Delta = 0$ so that $A = B = I_n$, and we conclude that $k^{-1}I_n$ is an extreme point.

The second assertion is trivial—we can assume $U = I_n$ and write

$$n^{-1}I_n = n^{-1}(I_n - E_{nn}) + n^{-1}E_{nn}$$

(E_{ij} is the matrix whose (i, j) entry is 1 and the remaining entries are 0). Then

$$\begin{aligned} \varphi_n(n^{-1}I_n) &= 1, \\ \varphi_n(n^{-1}(I_n - E_{nn})) &= (n-1)/n, \\ \varphi_n(n^{-1}E_{nn}) &= n^{-1}. \end{aligned}$$

Thus setting $A = n^{-1}(I_n - E_{nn})$, $B = n^{-1}E_{nn}$, we have

$$n^{-1}I_n = \varphi_n(A)\frac{A}{\varphi_n(A)} + \varphi_n(B)\frac{B}{\varphi_n(B)}$$

and $\varphi_n(A) + \varphi_n(B) = 1$. Thus $n^{-1}I_n$ is not extreme. ∎

We can complete the information in Theorem 3 by determining the extreme points of S_k for any value of k. Let $\mathscr{R}_k$ denote the set of $A \in M_n(\mathbb{C})$ such that $\rho(A) = 1$ and $\varphi_k(A) = 1$. Let $U_n(\mathbb{C})$ denote the unitary group in $M_n(\mathbb{C})$ so that $k^{-1}U_n(\mathbb{C})$ consists of all matrices $k^{-1}U$, $U \in U_n(\mathbb{C})$.

LEMMA 2. *If A is an extreme point of S_k then $A \in \mathscr{R}_k \cup k^{-1}U_n(\mathbb{C})$.*

Proof. Suppose on the contrary that A is an extreme point of S_k and A is neither a rank 1 matrix nor a multiple of a unitary matrix. We can assume that

$$A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$$

where $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ are the singular values of A (i.e., φ_k is invariant under unitary equivalence). Then the conditions that A be a point of S_k, not in $\mathscr{R}_k \cup k^{-1}U_n(\mathbb{C})$, imply

$$\alpha_1 + \cdots + \alpha_k = 1, \tag{13}$$

$$\rho(A) = m > 1, \tag{14}$$

$$\alpha_1 > \alpha_n. \tag{15}$$

Let $E_m = E_{11} + \cdots + E_{mm}$. We express A as a sum of matrices,

$$A = C + D, \tag{16}$$

in which C and D are linearly independent and

$$\varphi_k(C) + \varphi_k(D) = 1. \tag{17}$$

For, (16) and (17) can be combined into

$$A = \varphi_k(C)\frac{C}{\varphi_k(C)} + \varphi_k(D)\frac{D}{\varphi_k(D)},$$

a convex combination of linearly independent matrices of norm 1.

Case 1. $\alpha_1 > \alpha_m$. Set $C = \alpha_m E_m$, $D = A - \alpha_m E_m$. Then since $\alpha_1 > \alpha_m$, C and D are linearly independent,

$$\varphi_k(C) = \min\{m, k\}\alpha_m ,$$

and since $D = \operatorname{diag}(\alpha_1 - \alpha_m, \alpha_2 - \alpha_m, ..., \alpha_{m-1} - \alpha_m, \alpha_m - \alpha_m, 0, ..., 0)$,

$$\begin{aligned}\varphi_k(D) &= \varphi_k(A) - \varphi_k(C)\\ &= 1 - \varphi_k(C).\end{aligned}$$

Case 2. $\alpha_1 = \alpha_m$, $m \geqslant k$. Observe that (15) implies that $m < n$. Let $C = \frac{1}{2}(A + \alpha_m E_{nn}) = \frac{1}{2}\operatorname{diag}(\alpha_1, ..., \alpha_m, 0, ..., 0, \alpha_m)$, and $D = \frac{1}{2}(A - \alpha_m E_{nn}) = \frac{1}{2}\operatorname{diag}(\alpha_1, ..., \alpha_m, 0, ..., 0, -\alpha_m)$. Clearly $A = C + D$ and $\varphi_k(C) = \frac{1}{2}(\alpha_1 + \cdots + \alpha_k) = \frac{1}{2} = \varphi_k(D)$.

Case 3. $\alpha_1 = \alpha_m$, $m < k$. Let $C = \alpha_1 E_{11}$, $D = \alpha_1(E_{22} + \cdots + E_{mm})$. Then $A = C + D$ and $\varphi_k(C) = \alpha_1 = 1/m$, $\varphi_k(D) = (m-1)\alpha_1 = (m-1)/m$. ∎

LEMMA 3. *Let $A \in \mathscr{R}_k$. Then A is an extreme point of S_k iff $k > 1$.*

Proof. Again, since φ_k is unitarily invariant we may assume $A = E_{11}$. If $k = 1$, set $C = \frac{1}{2}(E_{11} + E_{22})$, $D = \frac{1}{2}(E_{11} - E_{22})$. Then $C + D = E_{11}$, $\varphi_1(C) = \frac{1}{2} = \varphi_1(D)$, and C and D are linearly independent. Thus A us not an extreme point of S_1.

Conversely, we show that if $k > 1$, then A is an extreme point of S_k. Suppose that

$$E_{11} = \tfrac{1}{2}(C + D), \tag{18}$$

where $\varphi_k(C) = \varphi_k(D) = \varphi_k(E_{11}) = 1$. Since $\rho(E_{11}) = 1$, $\varphi_1(E_{11}) = 1$ and (18) implies that

$$\begin{aligned}1 &= \varphi_1(E_{11})\\ &\leqslant \tfrac{1}{2}\varphi_1(C) + \tfrac{1}{2}\varphi_1(D)\\ &\leqslant \tfrac{1}{2}\varphi_k(C) + \tfrac{1}{2}\varphi_k(D)\\ &= 1.\end{aligned} \tag{19}$$

The inequality $\varphi_1(C) \leqslant \varphi_k(C)$ is strict (i.e., $k > 1$) unless $\rho(C) = 1$ so that (19) implies that $\rho(C) = \rho(D) = 1$, and hence $\varphi_{2,n}(C) = \varphi_k(C)$, $\varphi_{2,n}(D) = \varphi_k(D)$. Thus from (18) we have

$$\begin{aligned}1 &= \varphi_{2,n}(E_{11})\\ &\leqslant \tfrac{1}{2}\varphi_{2,n}(C) + \tfrac{1}{2}\varphi_{2,n}(D)\\ &= \tfrac{1}{2}\varphi_k(C) + \tfrac{1}{2}\varphi_k(D)\\ &= 1.\end{aligned} \tag{20}$$

The triangle inequality for the Euclidean norm $\varphi_{2,n}$ is strict in (20) unless $C = rD$, $r > 0$ (see the remarks following Lemma 1). But then $1 = \varphi_k(C) = r\varphi_k(D) = r$ and $C = D$. ∎

The following theorem shows that for every k the extreme points of S_k are either multiples of rank 1 or unitary matrices. More precisely, we have

THEOREM 4. *The extreme points of S_k are*:

(i) $U_n(\mathbb{C})$ *if* $k = 1$;

(ii) $\mathscr{R}_n$ *if* $k = n$;

(iii) $\mathscr{R}_k \cup k^{-1}U_n(\mathbb{C})$ *if* $1 < k < n$.

Proof. (i) Theorem 3 shows that any unitary matrix is an extreme point of S_1. Lemma 2 asserts that any extreme point of S_1 is either unitary or of rank 1 and Lemma 2 excludes the rank 1 possibility.

(ii) Theorem 3 states that no element of $n^{-1}U_n(\mathbb{C})$ is an extreme point of S_n. Combined with Lemma 2 this implies that only the elements of $\mathscr{R}_n$ are candidates for extreme points of S_n and Lemma 3 asserts that every matrix in $\mathscr{R}_n$ is an extreme point.

(iii) For $1 < k < n$, Theorem 3 and Lemma 3 together imply that every matrix in $\mathscr{R}_k \cup K^{-1}U_n(\mathbb{C})$ is an extreme point of S_k and Lemma 2 asserts that there are no others. ∎

3. STRUCTURE OF ISOMETRIES

We prove the three parts of Theorem 1, $k = 1$, $k = n$, $1 < k < n$, separately.

k = 1. We have $\varphi_1(T(A)) = \varphi_1(A)$ for all $A \in M_n(\mathbb{C})$. Since T is an isometry it must map the extreme points in S_1 into such, so that by Theorem 4(i), $T(U_n(\mathbb{C})) \subset U_n(\mathbb{C})$. In [4] it is proved that any T preserving the unitary group must have the form (1). Thus the Morita result [9] is proved.

k = n. In this case Theorem 4(ii) implies that $T(\mathscr{R}_n) \subset \mathscr{R}_n$. Thus T maps the set of rank 1 matrices in $M_n(\mathbb{C})$ into itself. The structure of such T was determined in [7] where it was proved that

$$T(A) = PAQ, \qquad A \in M_n(\mathbb{C}), \tag{21}$$

or

$$T(A) = PA'Q, \qquad A \in M_n(\mathbb{C}), \tag{22}$$

for fixed nonsingular P and Q. Let the singular values of P and Q be

$\alpha_1 \geqslant \cdots \geqslant \alpha_n$ and $\beta_1 \geqslant \cdots \geqslant \beta_n$, respectively. Then $P = U_1 D_1 V_1$, $Q = U_2 D_2 V_2$ where U_i, $V_i \in U_n(\mathbb{C})$ and $D_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, $D_2 = \mathrm{diag}(\beta_1, \ldots, \beta_n)$. Then in case (21) holds,

$$\begin{aligned}\varphi_n(A) &= \varphi_n(V_1{}^* A U_2{}^*)\\ &= \varphi_n(T(V_1{}^* A U_2{}^*))\\ &= \varphi_n(U_1 D_1 V_1 V_1{}^* A U_2{}^* U_2 D_2 V_2)\\ &= \varphi_n(D_1 A D_2), \qquad A \in M_n(\mathbb{C}). \end{aligned} \tag{23}$$

If $A = E_{11}$ then (23) implies that $1 = \varphi_n(E_{11}) = \varphi_n(\alpha_1\beta_1 E_{11}) = \alpha_1\beta_1$. Similarly if $A = E_{nn}$ then $1 = \alpha_n\beta_n$. It follows from the ordering of the α_i and β_i that $\alpha_1 = \cdots = \alpha_n$, $\beta_1 = \cdots = \beta_n$. Hence $P = \alpha U$, $Q = (1/\alpha)V$, $U, V \in U_n(\mathbb{C})$ and $T(A) = UAV$. The proof that T has the form (1) in case (22) holds is identical. This proves Russo's result [11]. It is interesting to note that in [11] the structure of T is made to depend on the form of linear maps that hold the unitary group invariant rather than the set of rank 1 matrices.

$1 < k < n$. Theorem 4(iii) implies that if $\rho(A) = 1$ or $A \in U_n(\mathbb{C})$ then $\rho(T(A)) = 1$ or $T(A)$ is a multiple of a unitary matrix. We show in fact that if $\rho(A) = 1$, then $\rho(T(A)) = 1$ and then apply the result in [7] to conclude that T has the form (1). Thus suppose $\rho(A) = 1$ but $T(A) = cU$, $U \in U_n(\mathbb{C})$. Since φ_k is unitarily invariant we may assume $A = E_{11}$. Now $\rho(sE_{11} + tE_{12}) = 1$ for any s and t not both 0. Hence

$$T(sE_{11} + tE_{12}) = scU + tT(E_{12}) \tag{24}$$

is either rank 1 or a multiple of a unitary matrix. There are also two possibilities for $T(E_{12})$, again, either rank 1 or a multiple of a unitary matrix. Note also that since E_{11} and E_{12} are linearly independent, U and $T(E_{12})$ are linearly independent. Suppose first that $T(E_{12}) = B$, $\rho(B) = 1$. If we multiply (24) by U^{-1} neither its rank nor the fact that it is a multiple of a unitary matrix is affected. Thus the matrix

$$scI_n + tU^{-1}B \tag{25}$$

either has rank 1 or is a multiple of a unitary matrix for all s, t not both 0. This is obvious nonsense and we conclude that $T(E_{12}) = B$ must be unitary. But once again (25) cannot be a multiple of a unitary matrix because U and B are linearly independent, nor does it have rank 1 for all s, t not both 0. Thus we conclude that $\rho(A) = 1$ implies $\rho(T(A)) = 1$ so that by [7], T has the form (21) or (22). By an argument similar to the case $k = n$ we see that T has the required form, i.e., P and Q may be taken to be unitary. ∎

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